Multifractal Interpolation of Universal Multifractals

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Abstract

Basing on invariant properties of universal multifractals we propose a simple algorithm for interpolation of multifractal densities. This algorithm admits generalizations to a multidimensional case. Analytically obtained are multifractal characteristic of the function interpolating initial data. We establish the relation between the parameter existing in algorithm and the Levy index α which is the main index for scaling function of universal multifractals

1 Introduction

Construction of the models for complex highly variable natural processes is closely associated with the study of their multifractal characteristics. Popular methods of averaging allows one to study smoothed characteristics only and lead to the loss of information about variability of the processes on scales less than the scales of averaging. Therefore, the problem is how to construct the function averaged over the scales $\epsilon << \epsilon_0$ in the limit $\epsilon \to 0$ with the help of smoothed experimental data (that is, averaged over some given scale ϵ_0). In fact, it implies the reconstruction of variability at small scales. Obviously, this problem has no unique solution without additional requirements and principles of such an interpolation. Examples of fractal interpolations with different principles are presented, for example, in Refs. [1], [2]. In the paper we use other principles which are

related to the possibility to return from interpolating function to the smoothed one by averaging over scales. At the same time, returning from an arbitrary interpolation step (that is, from the interpolating function on an arbitrarily small scale) we should get an initial function. The second principle is associated with the idea of similarity in a broad sense, and implies the conservation of memory about small-scale variability of unknown strict function in smoothed data. Similar principles are already widely used in various cascade models of turbulence (see, e.g.α-model [3], [4], and β-model [5], [6]). Besides, we use a natural requirement that interpolated (smoothed) function possesses definite multifractal characteristics. This requirement is general for all methods of fractal [1], [2] and multifractal interpolation. We use the momentum scaling function $K(q)$ as a multifractal characteristic. It is related by the known way with the spectrum of singularity $f(\alpha)$ [7], [8], [9] and the mass power [10], [11]. At least, we require the simplicity of the interpolation procedure, this requirement being hardly defined. We propose a simple and fast interpolation algorithm which allows one to make its generalization on a multidimensional case. The multifractal characteristics of the function interpolating initial data are obtained analytically. It is shown, that multifractals arising as a result of interpolation, belongs to universal multifractals [12] associated with the deep properties of stochastic processes. The proposed interpolation method is based on the existence of the invariants (that is, the characteristics independent of scales) obtained in the paper for the universal multifractals.

2 Formulation of the problem and the algorithm

Let us consider the process of smoothing the function $\rho(x)$. We assume without loss of generality that the unit interval [0,1] is the definition region of this function. Then, $\rho(x)$ is assumed as integrable. We divide the definition region in to N equal segments $[0,1] = \bigcup_{n=1}^{N} I_n$, $I_n = ((n-1)\epsilon, n\epsilon)$ where $\epsilon = 1/N$. We defined the step-like function $\rho_\epsilon(x)$ which has a non-zero constant value at each interval I_n,

$$\rho_\epsilon(x) = \frac{\int\limits_{n\epsilon}^{(n+1)\epsilon} \rho(x)dx}{\epsilon},\ x \in I_n \tag{1}$$

Where $x \in [0,1]$. The function $\rho_\epsilon(x)$ smoothes the function $\rho(x)$ at scale ϵ . Successive increasing the scales of smoothing allows us pass to more and more smoothed description of $\rho(x)$. The function $\rho_1(x)$ (L=1) is the last in the hierarchy of smoothed descriptions,

$$\rho_1(x) = \frac{\int\limits_0^L \rho(x)dx}{L} \equiv \langle\rho\rangle \tag{2}$$

where $\langle ... \rangle$ the parentheses imply spatial averaging. The function ρ_ϵ, $x \in [0,1]$ form a completely ordered set. It is easily proved that $\langle ... \rangle$ is invariant at all stages of smoothing. Indeed,

$$\langle \rho_\epsilon(x) \rangle = \frac{\int\limits_0^L \rho_\epsilon(x)dx}{L} = \frac{\sum\limits_{i=0}^{N-1} \int\limits_{i\epsilon}^{(i+1)\epsilon} \rho(x)dx\epsilon}{L\epsilon} =$$
$$= \frac{1}{L} \sum_{i=0}^{N-1} \int\limits_{i\epsilon}^{(i+1)\epsilon} \rho(x)dx = \frac{1}{L} \int\limits_0^L \rho(x)dx = \langle \rho \rangle \tag{3}$$

Therefore, the mean value of any smoothed function $\rho_\epsilon(x)$ coincides with that of the initial function. The described process of smoothing simulated the main features of experimental data treatment. Let us formulate the problem. We assume that the initial density $\rho(x)$ is a multifractal with given momentum scaling function $K(q)$. Knowing smoothed description $\rho_{\epsilon_0}(x)$, we try to restore $\rho_\epsilon(x)$ at $\epsilon << \epsilon_0$ with the same momentum function. In other words, we create a model of the function$\rho(x)$ with conserved properties of variability at small scales $\epsilon \to 0$. The model $\tilde{\rho}(x) = \lim\limits_{\epsilon \to 0} \rho_\epsilon(x)$ is named as interpolating function of the smoothed density ρ_{ϵ_0} . According to the meaning of smoothing, it is not necessary for $\tilde{\rho}(x)$ to get the fixed values which coincide with the corresponding values $\rho_{\epsilon_0}(x)$ of at the definite x. We expect that this problem has no unique solution, because the integration procedure in Eq.(1) implies the loss of information about the variability of $\rho(x)$ on scales less than ϵ . Besides, after the reconstruction of its variability, the interchanges of particular segments of the function are assumed, those do not influence their multifractal characteristics. Therefore, let us formulate the particular algorithm of interpolation and discuss its properties. We assume that given smoothed step-like function $\rho_{\frac{1}{p}}(x)$ (p is some integer) has some value not equal zero at each interval $\frac{1}{p}$ long. The absence of zero values is not principal and is assumed for the simplicity of subsequent analysis only. Therefore, we know $\rho_\epsilon(x)$ at $\epsilon > \frac{1}{p}$, and, in particular, $\rho_1 = \langle \rho \rangle$. Below we assume $\langle \rho \rangle$ without loss of generality. We describe the algorithm as successive iterations of $\rho_{\frac{1}{p}}(x)$, each of them consisting in 3 simple operations:

1. The contraction of from the interval [0,1] onto the intervals $[0,\frac{1}{p^2}]$, $[\frac{1}{p^2}, \frac{2}{p^2}]$, $[\frac{p^2-1}{p^2}, 1]$ and the construction of the function with the period $\frac{1}{p^2}$. The periodic function repeats on its period the behavior of $\rho_{\frac{1}{p}}(x)$ at the whole interval [0,1], that is it coincides with $\rho_{\frac{1}{p}}(xp^2)$ (operation $\mathbf{P}$).

2. The periodic function obtained at the 1-st step is raised to the ν -th power (operation $\mathbf{R}$). In general case the power ν can depend on the number n of the interval $I_n = [\frac{n-1}{p^2}, \frac{n}{p^2}]$, and on the iteration step. We restrict ourselves to the case ν=const, for simplicity.

3. The mean value is normalized (operation **N**), that is, the mean value at each interval I_n is fitted to the corresponding value of $\rho_{\frac{1}{p}}(x)$ at the same interval. This is done by multiplication of the values at the interval I_n on the value of $\rho_{\frac{1}{p}}(x)$ at $x \in I_n$, and subsequent division of the function, constructed after the steps 1 and 2, on the mean value at this interval.

A single iteration step of the function $\rho_{\frac{1}{p}}(x)$ consists in applying operation **NRP** to it. An analogous transition is realized from the iteration step at the scale of smoothing ϵ_k to k+1 - st step at the scale $\epsilon_{(k+1)}$, that is,

$$\rho_{\epsilon_{k+1}}(x) = \mathbf{NRP}\rho_{\epsilon_k}(x) \tag{4}$$

It is seen, that during such an algorithm $\epsilon_{k+1} = \epsilon_k^3$ while the number of intervals $N_{k+1} = \frac{1}{\epsilon_{k+1}}$ is increased according to $N_{k+1} = N_k^3$. It implies that, if we have p intervals initially, then on the k-th step $N_k = p^{3^k}$, that is, the variability of the function $\rho_{\epsilon_k}(x)$ grows hyper exponentially. The operations **P,R** are based on scale similarity properties of $\rho(x)$ and conservation of "memory" during the smoothing. Besides, **R** has a control parameter ν influencing the multifractal properties of the interpolating function. The operation **N** ensures conservation of the mean values and is related to the invariant of smoothing described above. The formulated algorithm is exclusively simple, easily realizable on computers and allows one to provide a complete analytical analysis of the interpolating functions.

3 Analysis of multifractal properties of interpolating functions

Let us proceed with the study of the multifractal characteristics of the function $\rho_\epsilon(x)$, which are constructed according to the algorithm presented above. It is convenient to pass from $\rho_{\epsilon_0}(x)$ to the functions defined on integers. We give a natural ordinal number n to the segments of the unit segment subdivision with length ϵ_k, and associate the value of a step-like function $\rho_{\epsilon_k}(x)$ to the function $\rho_k(n)$ at this segment, that is,

$$\rho_k(n) = \rho_{\epsilon_k}(x \mid x \in [(n-1)\epsilon_k, n\epsilon_k]) \tag{5}$$

Then, the index k implies the number of iteration, while n=1,2,...,N_k. Here N_k is a number of the unit segment subdivision intervals with the length ϵ_k at the k-th iteration step. We formulate in an explicit form the rule of the transition (**NPR**) from $\rho_k(m)$ to

$\rho_{k+1}(n)$ (m=1,2,...,N_k; n=1,2,...,N_{k+1}). It is easily proved that

$$\rho_{k+1}(n) = \frac{{\rho_k}^{\nu}(n \bmod N_k)}{M_\nu(k)}\rho_k(1+\left[\frac{n}{{N_k}^2}\right]) \tag{6}$$

Here the standard notations are used; the square brackets denote an integer part of a number, (j + p N_k) mod N_k =j if, $0 \leq j \leq N_k$ while p is an integer;n=1,..,N_k. $M_\nu(k)$ is a ν -th moment of $\rho_{\epsilon_k}(x)$, that is,

$$M_\nu(k) = \frac{\int_0^1 \rho_{\epsilon_k}(x)dx}{1} = \epsilon_k \sum_{n-1}^{N_k} {\rho_k}^{\nu}(n) = \frac{1}{N_k}\sum_{n=1}^{N_k} {\rho_k}^{\nu}(n) \tag{7}$$

Eq. (6) allows us to calculate easily arbitrary moments of the introduced density $\rho_k(n)$. As an example, we prove the conservation of the mean density on an arbitrary iteration step. By definition, the mean density at (k+1)-st iteration step is

$$M_1(k+1) =< \rho_{k+1} >= \frac{1}{N_{k+1}}\sum_{n=1}^{N_{k+1}} \rho_{k+1}(n)$$

Inserting the expression for $\rho_{k+1}(n)$ see Eq. (6), and taking into account that $N_{k+1} = N_k^3$, after partial summation we get

$$< \rho_{k+1} >= \frac{1}{{N_k}^3} N_k \frac{\sum_{n=1}^{N_k} {\rho_k}^{\nu}(n)}{M_\nu(k)} \sum_{n=1}^{N_k} \rho_k(n) = \frac{M_\nu(k)}{M_\nu(k)}\frac{1}{N_k}\sum_{n=1}^{N_k}\rho_k(n) \equiv< \rho_k >$$

Therefore, when using the described algorithm, the mean value is conserved at arbitrary iteration step. We assume ¡ ρ¿ = 1 without loss of generality. Then, we determine how multifractal characteristics of the density $\rho_k(n)$ (that is, those of $\rho_{\epsilon_k}(x)$) depend on the free parameter ν. For this purpose it is necessary to calculate more general fractional order moments of the constructed function $\rho_k(n)$. At the first step, using Eq.(6), we establish the relation between the q-th moment at the (k+1)-st iteration step with that at the k-th iteration step. By definition

$$M_q(k+1) = \frac{1}{N_{k+1}}\sum_{n=1}^{N_{k+1}} {\rho_{k+1}}^q(n) \tag{8}$$

Using Eq. (6) and $N_{k+1} = N_k^3$, after partial summation we get

$$M_q(k+1) = \frac{M_{q\nu}(k)}{{M_\nu}^q(k)} M_q(k) \tag{9}$$

Here the standard notations of the moments are used, see Eqs. (7),(8). We use the momentum formalism [12], [13] in order to determine multifractal characteristics of the interpolating function. The main definition of this formalism is that of the momentum scaling function $K(q)$ of multifractal density $\rho_\epsilon(x)$ at scale resolution ϵ:

$$< {\rho_\epsilon}^q(x) >= \epsilon^{-K(q)} \tag{10}$$

The function $K(q)$ is obeyed the conditions $K(0) = K(1) = 0$ which are the consequences of the definition (10) and the relation ¡ ρ ¿ $= 1$. It is worthwhile to note that $K(q)$ is independent of scales , and related by the known way to the mass power $\tau(q)$ [14]:

$$\tau(q) = 1 - q + K(q)$$

The mass power is related by a Legendre transform to the singularity function $f(\alpha)$, see for example, Ref [14]. Returning to Eq.(9), we consider it as a functional equation for the moments of interpolating function. We look for the solution in the form

$$M_q(k) = \epsilon_k^{-K(q)} \tag{11}$$

Then, for the scaling function K(q) we get the equation

$$\epsilon_{k+1}^{-K(q)} = \epsilon_k^{qK(\nu)-K(q\nu)-K(q)}$$

Using $\epsilon_{k+1} = \epsilon_k^3$, it is easy to exclude the dependence on scales and get the functional equation for $K(q)$;

$$2K(q) = K(q\nu) - qK(\nu) \tag{12}$$

The absence of scale dependence in Eq.(13) implies that we really get multifractal density $\rho(x)$ as a result of interpolation. To find the solution of Eq. (13) we make a substation $K(q) = \psi(q)$ and get the equation for $\psi(q)$:

$$2\psi(q) = \nu(\psi(q\nu) - \psi(\nu))$$

We look for the solution in the class of differentiated functions. We differentiate this equation over q and get the following equation for the derivative $\psi'(q)$:

$$2\psi'(q) = \nu^2\psi'(q\nu)$$

The solution at $q > 0$ is

$$\psi'(q) = K_0 q^{\frac{\ln 2}{\ln \nu}-2}$$

Integrating this equation and returning to $K(q)$ we get ultimately

$$K(q) = \begin{cases} \frac{K_0}{\frac{\ln 2}{\ln \nu}-1} q^{\frac{\ln 2}{\ln \nu}} + c_1 q, & \text{if } \nu \neq 2 \\ qK_0 \ln q + c_1 q, & \text{if } \nu = 2 \end{cases}$$

Using natural boundary conditions $K(0) = K(1) = 0$, we exclude an arbitrary constant c_1:

$$K(q) = \begin{cases} \frac{K_0}{\alpha-1}(q^{\alpha} - q), & \text{if } \nu \neq 2 \\ K_0 q \ln q, & \text{if } \nu = 2 \end{cases} \tag{13}$$

where $\alpha = \frac{\ln 2}{\ln \nu}$, K_0 is an arbitrary constant. It is worthwhile to note that Eq. (13) coincides with the known $K(q)$ for the so-called universal multifractals [12], [13]. Extraction of this class has a deep mathematical origin associated with the existence of limit theorems for the distributions of the sums of independent stochastic quantities [15]. From the other hand, just this class of multifractals is the main pretender to the description of real natural processes and objects. This fact is supported by experimental investigations of various natural phenomena [12], [14].

Therefore, we have shown that interpolating function $\rho_\epsilon(x)$ is universal multifractal, the main characteristic $\alpha = \frac{\ln 2}{\ln \alpha}$ of it being the known function of the control parameter ν. Varying ν, it is possible to construct various interpolations of experimental data with various multifractal characteristics. It is worthwhile to note that this analysis, actually points to the existence of 2-parametric family of multifractal density invariants. In fact, it is easy to prove the following theorem: If there exist a and b such that

$$\frac{< {\rho_\epsilon}^{aq}(x) >}{< {\rho_\epsilon}^{a}(x) >^{q} < {\rho_\epsilon}^{q}(x) >^{b}} = const \tag{14}$$

is independent on scale at $q \in R$, then $\rho_\epsilon(x)$ is a multifractal density of universal multifractal with the Levy index $\alpha = \frac{\ln b}{\ln a}$. The proof of this theorem can easily be obtained if one consider Eq. (14) as a functional equation for the momentum scaling function and repeat the solution procedure this equation in a way analogous to that sited above. Inverse theorem is also valid, its proof being trivial after substitution of $K(q)$ for universal multifractals. This, the proposed scheme of multifractal density interpolation is based on the existence of the invariants (14) (independent of scale resolution) for universal multifractal distributions. Such invariants can be useful for the study of multifractal properties of the turbulence and of other strongly non equilibrium systems, because these invariants establish relation between the moments of different orders and determine splitting the moments for universal multifractals.

In conclusion we discuss convergence of the proposed algorithm. For this purpose it is convenient to rewrite Eq.(9) for the scaling function. The rewritten equation describes the scaling function evolution during the iteration process,

$$K_{n+1}(q) = \frac{1}{2}(K_n(q\nu) - qK_n(\nu)) \tag{15}$$

where $K_{n+1}(q)$ is the scaling function at the $(n+1)$-th iteration step. With the use of Eq.(15) we consider the scaling function at the n-th iteration step with the initial condition taken into account.

$$K_1(q) = \frac{K_0}{\alpha - 1}(q^\alpha - q) + \Delta(q) \tag{16}$$

The function $\Delta(q)$ is caused by initial deviation from the scaling function of the universal multifractal. Then, after n iterations we get

$$K_n(q) = \frac{K_0}{\alpha - 1}(\frac{\nu^{\alpha n}}{2^n})(q^\alpha - q) + \frac{1}{2^n}(\Delta(q\nu^n) - q\Delta(\nu^n)) \tag{17}$$

It follows from Eq.(17) that for $\nu^\alpha = 2$

$$K_n(q) \overset{n\to\infty}{\longrightarrow} \frac{K_0}{\alpha - 1}(q^\alpha - q) \tag{18}$$

if $\Delta(q)$ is bounded above and below (i.e. $C_1 < \Delta(q) < C_2$). Moreover, if $\Delta(q)$ grows not faster than constq^β with q growing, where $\beta < \alpha$, and $\beta = 1$, is also included, then the scaling function also converges to the scaling function of universal multifractal. Thus, it is proved that the moments of the interpolated multifractal converge to those of the universal multifractal (of course, if initial deviations do not grow rather fast with q drawing). It implies that the proposed algorithm can be used for interpolation of the universal multifractals. It is noteworthy that the simplicity of this algorithm also allows one to the use it effectively for studying properties of the universal multifractals by purely analytical methods. Therefore, the proposed algorithm allows one to interpolate experimental data and reconstruct multifractal interpolating functions, which belong to universal multifractals, with arbitrary index α. It allows one to use the algorithm for the description and creation of the models of various natural processes and structures, in particular, ecologically important distributions of technogenic admixtures precipitated after the transport in turbulent atmosphere.

Results

One - parameter family of invariants of universal multifractals has been obtained. The existence of such invariants implies the existence of rules allowing one to express higher one-point moments in terms of lower ones.

If for a multifractal one finds one of these invariants,this signifies that such a multifractal is in class of the universal multifractals with a definite Levy index. A simple algorithm for interpolating universal multifractals based on the invariance discovered has been proposed. The convergence of the algorithm has been proved, and velocity of convergence has been analytically estimated.

Acknowledgements

This work was supported by International Association under the project INTAS 93-1194.

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